Midsemester Examination - Algebra I

20th September, 2007

Attempt all questions
Total Marks 50

1. Let $\phi : G \to G'$ be a surjective homomorphism of groups. Let N be a normal subgroup of G. Prove that $\phi(N)$ is a normal subgroup of G'. Is this true if ϕ is not surjective? Justify your answer. (6+4 marks)

2. Let G be a group and define the centre, $Z(G)$, of G as:
$$Z(G) := \{a \in G | ga = ag \ \forall \ g \in G\}.$$
Prove that $Z(G)$ is a normal subgroup of G. Suppose that the quotient group $G/Z(G)$ is a cyclic group. Then prove that G is an abelian group. (3+7 marks)

3. Determine the group of automorphisms of a cyclic group of order 10. (10 marks)

4. Let G be the set of all real 2×2 matrices $\begin{pmatrix} a & b \\ 0 & d \end{pmatrix}$ where $ad \neq 0$, under matrix multiplication. Let $N = \left\{ \begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix} \right\}$. Prove that N is a normal subgroup of G and G/N is abelian. (4+6 marks)

5. Let G be the group $\mathbb{C}^*$ under multiplication. Consider the subgroup $H := \{1, -1, i, -i\}$ of G. Prove that the quotient group G/H is isomorphic to G. (10 marks)